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Three-Body Scattering Amplitude. I. Separation of Angular Momentum*

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The angular momentum is separated in the Fadeev equations for the three-body scattering amplitude. The method used is symmetrical with respect to the three particles and does not introduce any relative angular momentum of two particles. The resulting equations are well suited to a numerical solution and can be applied to a study of the problem of overlapping resonances. They also provide a natural starting point for an extension of the three-body scattering amplitude to complex angular momentum.

I. INTRODUCTION

THIS is the first of a series of papers devoted to the angular momentum analysis of the three-body scattering amplitude, by which we mean the amplitude for three particles going from one initial configuration to a final one through the effect of two-particle interactions. In the following paper, we shall see how one can find equations for the three-body scattering amplitude when the total angular momentum is made complex. The present article therefore treats of more immediate—and less mathematical—questions. The reason for the splitting is our hope that this work will prove useful for people who are interested in the extension of the Regge theory to three-body problems or who do not want to enter into the necessarily more sophisticated mathematics it involves.

When one tries to extend to three-body scattering, the well-known methods for treating two-body scattering, several new problems appear, beside the obvious increase in complexity due to the larger number of parameters.[1] Certainly the most essential such problem is the nonconnectedness of the scattering matrix. This means that processes are possible in which two of the particles interact while the third one has no interaction with them. In terms of perturbation-theory graphs, such a disconnected process is represented by a graph in which the propagation line for the third particle is disconnected from the rest of the graph (which represents the interaction of the two first particles). This phenomenon leads to difficulties which can be expressed in several ways:

If one uses the Lippman-Schwinger[2] equation to formulate the three-body scattering problem, the disconnectedness leads to the appearance of δ functions in the kernel. This in turn entails that the Lippman-Schwinger kernel has a continuous spectrum, which means that any classical approximation method such as, for instance, the iteration method, will converge very slowly if at all, and that it will be practically impossible to derive any analytic property of the solution as a function of the parameters such as the energy or the angular momentum. Another difficulty of the Lippman-Schwinger equation is that the homogeneous equation has solutions when there exist bound states of a pair of particles. Such a solution is in fact provided by the wave function for a scattering process in which the initial state contains a bound state. In fact, these two difficulties are linked, since one can derive the existence of a continuous spectrum from the existence of these solutions of the homogeneous equation. Generally, it is necessary to add to a solution of the complete Lippman-Schwinger equation a solution of the homogeneous equation in order to fit the boundary conditions, which means that the equation is not in fact very useful in practice.

This difficulty has been removed by Fadeev,[3] who has given a set of equations for the three-body scattering amplitude where there are no δ functions and whose kernel has no continuous spectrum. Another set of equations that can be extended to more than three-body processes has also been given by Weinberg.[4] In the following, we shall use the Fadeev equations, but most of what we shall say will also be valid for the Weinberg equations.

Another nontriviality of the three-body problem, although less fundamental than the preceding one, appears when one wants to separate the total angular momentum. The customary method[5] consists in introducing the relative angular momentum of particles 1 and 2, for instance, in their center-of-mass system and combining with the angular momentum of particle 3 in the total center-of-mass system in order to get the total angular momentum (here and in the following, we assume for simplicity that the three particles are spinless). This procedure leads to very slowly convergent expressions when one wants to consider states of particles 1 and 3 together, as in rearrangement collisions. We shall show how this difficulty can be removed by never introducing the relative angular momentum of a pair of particles.

Our method consists in associating a reference system

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[1] General references concerning three-particle scattering: H. Ekstein, Phys. Rev. **101**, 880 (1956); T. F. Jordan, J. Math. Phys. **3**, 429 (1962); A. G. Tixaire, Helv. Phys. Acta **32**, 412 (1959); G. Grauert and J. Petzold, Z. Naturforsch. **15a**, 311 (1960).

[2] B. A. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[3] L. D. Fadeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[4] Steven Weinberg, Phys. Rev. **133**, B232 (1964).

[5] R. G. Newton, Ann. Phys. 4, 29 (1958).

of axes in the momentum representation with each configuration of the three particles. The wave function is therefore a function of the total momentum, the three energies of the particles in the total center-of-mass system, and three Euler angles which characterize the position of the body-fixed axes. Correspondingly, the quantum numbers are the total momentum, the three energies, the total angular momentum and its two projections on a body-fixed axis and on a space-fixed axis. Because of the conservation of angular momentum, its projection on the space-fixed axis will be a constant that does not appear in the final equations. This method preserves the symmetry of the problem with respect to the three particles. An unexpected result is that the Fadeev equations then assume a simple enough form that it is a fair hope to solve them on a computer. We thus get a method for investigating the important problems of three-body resonances and of the interference of several resonances in a Dalitz plot.

The interest of this extremely simple technique can best be seen by comparing it with the present studies of three-body scattering problem with separation of the angular momentum.[5,6] To our knowledge, they consist in introducing the relative angular momentum of particles 1 and 2, as said before. Then the total wave function is projected out on a complete set of states of particles 1 and 2; these states are solutions of the Schrödinger equation for these particles interacting through their mutual potential. These two-particle states are characterized by the angular momentum l_{12} and the energy E_{12}, which is an eigenvalue of the two-particle Hamiltonian. The Schrödinger equation for the three-body wave function, after separation of the total angular momentum, appears then as a differential equation where the only variable is the distance between particle 3 and the center of mass of particles 1 and 2. This equation is formally very similar to the Schrödinger equation of a two-body problem, and looks extremely simple. Unfortunately, this simplicity is only apparent. The first difficulty is that the wave function is now a matrix with indices l_{12}', E_{12}', initial values of the parameters, and l_{12}, E_{12}, projection indices. As E_{12} and E_{12}' are continuous parameters, this means that one is in fact dealing with matrices with continuous indices, which makes any correct mathematical analysis very difficult.

This method has been used by Newton[6] to investigate the properties of the three-body scattering amplitude as a function of the angular momentum.[7]

In Sec. II we recall rapidly the Fadeev equations (which are apparently not so widely well known as they should be). This will also fix the notations. The wave functions that allow symmetric reduction of the total angular momentum are introduced in Sec. III. In Sec. IV, we analyze in detail the reduction of the inhomogeneous term of the Fadeev equation; the complete equations are considered in Sec. V together with a discussion of the possible applications of the results.

II. THE FADEEV EQUATIONS

Let us consider three nonrelativistic spinless particles with masses m_1, m_2, m_3. They will be assumed to be different. The Hamiltonian of the system has the form

$$H=T_1+T_2+T_3+V_{23}+V_{31}+V_{12}, \qquad (1)$$

where $T_i=(1/2m_i)\nabla_i^2$ and V_{ij} is a two-body potential acting on the variable $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$ of the wave function.

It will be convenient to introduce the two-body scattering amplitudes. The Hamiltonian of the system made up by the two particles 1 and 2 is

$$\hat{H}=\hat{T}_1+\hat{T}_2+\hat{V}_{12}, \qquad (2)$$

and the scattering matrix for these two particles is defined by the Lippman-Schwinger equation,

$$\hat{T}_{23}(z)=\hat{V}_{23}-\hat{V}_{23}(\hat{T}_2+\hat{T}_3-z)^{-1}\hat{T}_{23}(z). \qquad (3)$$

Here the "hat" means that the operators act in the Hilbert space of two-body states, and we have explicitly introduced the parameter z, which indicates an extension off the shell of real energies. The scattering matrix for physical two-body scattering is defined as

$$\hat{T}_{23}=\lim_{z\to E+i0}\hat{T}_{23}(z), \qquad (4)$$

E being the total energy (for instance, the total initial energy).

One can also define the scattering matrix for two-body processes in the Hilbert state of three-body states as the scattering matrix in the absence of interactions between particles 1 and the two other particles, i.e., by

$$T_{23}(z)=V_{23}-V_{23}G_0(z)T_{23}(z), \qquad (5)$$

where $G_0(z)$ is the Green's function $(T_1+T_2+T_3-z)^{-1}$. Using these definitions, one gets immediately the relation between T_{23} and $\hat{T}_{23}$ as

$$\langle \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|T_{23}(z)|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle =\langle \mathbf{p}_1'|\mathbf{p}_1\rangle\langle \mathbf{p}_2',\mathbf{p}_3'|\hat{T}_{23}(z-p_1^2/2m_1)|\mathbf{p}_2,p_3\rangle. \qquad (6)$$

Fadeev has shown that the amplitude for a transition between an initial configuration of the three free particles with momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and a final configuration with momenta $\mathbf{p}_1'$, $\mathbf{p}_2'$, $\mathbf{p}_3'$ is the matrix element of the scattering matrix T,

$$\lim_{z\to E+i0}\langle \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|T_{(z)}|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle, \qquad (7)$$

where T can be written as a sum of three contributions,

$$T(z)=T(z)^{(1)}+T(z)^{(2)}+T(z)^{(3)}, \qquad (8)$$

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[6] R. G. Newton, Nuovo Cimento **24**, 400 (1963); Phys. Letters **4**, 11 (1963); J. B. Hartle (to be published).
[7] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).

which satisfy a set of equations that, in a matrix form, is

$$\begin{bmatrix} T^{(1)}(z) \\ T^{(2)}(z) \\ T^{(3)}(z) \end{bmatrix} = \begin{bmatrix} T_{23}(z) \\ T_{31}(z) \\ T_{12}(z) \end{bmatrix} - \begin{bmatrix} 0 & T_{23}(z) & T_{23}(z) \\ T_{31}(z) & 0 & T_{31}(z) \\ T_{12}(z) & T_{12}(z) & 0 \end{bmatrix} G_0(z) \begin{bmatrix} T^{(1)}(z) \\ T^{(2)}(z) \\ T^{(3)}(z) \end{bmatrix}. \tag{9}$$

It is very easy to see the meaning of Eq. (9) in terms of graphs of perturbation theory: Let us call $T^{(1)}(z)$ the sum of the contributions of the set of all graphs where the last interaction is between particles 2 and 3 through the potential V_{23}. Clearly, $T^{(1)}(z)$ contains the contributions from all graphs where particles 2 and 3 interact any number of times without interacting with particle 1, i.e., it contains $T_{23}(z)$. All other contributions to $T^{(1)}(z)$ are from graphs where particles 2 and 3 interact any number of times before particles 1 and 3, for instance, interact through potential V_{13} and then anything else happens. This gives the contribution $-T_{23}(z)G_0(z)T^{(2)}(z)$ of the Fadeev equations. The important property of these equations is that, if we call $K(z)$ the Fadeev's kernel,

$$K(z) = \begin{bmatrix} 0 & T_{23}(z) & T_{23}(z) \\ T_{31}(z) & 0 & T_{31}(z) \\ T_{12}(z) & T_{12}(z) & 0 \end{bmatrix} G(z), \tag{10}$$

while the matrix elements of $K(z)$ contain δ functions of the momenta, the square $K^2(z)$ (in the operator sense) does not contain any δ functions, owing to the zeros on the main diagonal. Furthermore, as has been proved by Fadeev[8] and Lovelace,[9] $K^2(z)$ is completely continuous. This means essentially that

$$\mathrm{Trace}K^2(z)K^2(z)^\dagger < \infty, \tag{11}$$

if the potentials V_{ij} are superposition of Yukawa potentials,

$$V_{ij}(r_{ij}) = \int_\mu^\infty d\sigma_{ij}(\mu)(e^{-\mu r_{ij}}/r_{ij}), \tag{12}$$

and if the nondecreasing function $\sigma_{ij}(\mu)$ is of bounded variation. Equation (11) is true for values of z which are not equal to the energy of a physical state. As, furthermore, $K^2(z)$ is a bounded kernel in that case, it follows as shown explicitly by Lovelace,[9] that the scattering matrix T is an analytic function of z except for these values, which are equal to the energy of a physical state.

These equations need to be completed when one wants to include initial or final states that contain bound states of a pair of particles. We shall not enter into these refinements here, just assuming that we are dealing with a case in which there are no such bound states. The modifications that have to be introduced when this hypothesis is abandoned will be considered if necessary in a later paper on practical applications.

[8] L. D. Fadeev, Dokl. Akad. Nauk SSSR **138**, 565 (1961) and **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961) and **7**, 600 (1963)].

[9] C. Lovelace, Lectures at Edinburgh Summer School, 1963 (unpublished).

III. A SET OF WAVE FUNCTIONS

Let us now consider the kinematics of the three-body system. A state can be characterized by the three momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. The corresponding kinetic energies ω_1, ω_2, ω_3 are equal to

$$\omega_i = p_i^2/2m_i. \tag{13}$$

The total momentum is $\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3$. We shall need also to introduce the relative momentum of particles 2 and 3 in their relative center-of-mass system as

$$\mathbf{q}_{23} = (m_3\mathbf{p}_2 - m_2\mathbf{p}_3)/(m_2+m_3). \tag{14}$$

We shall occasionally use a special notation for certain sums of masses like $m_{23}=m_2+m_3$ and $M=m_1+m_2+m_3$. The reduced mass of particles 2 and 3 in their relative center-of-mass system is m_2m_3/m_{23}, and their energy in that system is

$$E_{23} = m_{23}\mathbf{q}_{23}^2(2m_2m_3)^{-1}. \tag{15}$$

We shall need frequently the angle θ_{23} between the two momenta $\mathbf{p}_2$ and $\mathbf{p}_3$ in the total center-of-mass system. Its cosine is given by

$$\cos\theta_{23} = \frac{p_2^2+p_3^2-p_1^2}{2p_2p_3} = \frac{m_2\omega_2+m_3\omega_3-m_1\omega_1}{2(m_2m_3\omega_2\omega_3)^{1/2}}. \tag{16}$$

We shall also need to introduce the angle γ_1 between $\mathbf{p}_1$ in the total center-of-mass system and $\mathbf{q}_{23}$ in the relative center-of-mass system of particles 2 and 3. It is defined through

$$\mathbf{p}_1\cdot\mathbf{q}_{23} = 2\frac{m_{23}(m_3\omega_3-m_2\omega_2)+(m_2-m_3)m_1\omega_1}{m_{23}}; \tag{17}$$

one easily deduces the value of γ_1 through

$$\cos\gamma_1 = (\mathbf{p}_1\cdot\mathbf{q}_{23})p_1^{-1}q_{23}^{-1}, \tag{18}$$

where

$$\mathbf{q}_{23}^2 = 2m_2m_3m_{23}^{-1}(\omega_1+\omega_2+\omega_3) - 2m_2m_3Mm_{23}^{-2}\omega_1. \tag{19}$$

This kinematics being rather cumbersome, it is often convenient to consider the special case $m_1=m_2=m_3=1$ where the formulas simplify greatly. One has, in that case,

$$p_1^2 = 2\omega_1, \tag{19a}$$

$$\cos\theta_{23} = (\omega_2+\omega_3-\omega_1)(4\omega_2\omega_3)^{-1/2}, \tag{19b}$$

$$q_{23}{}^2=\tfrac{1}{2}(2\omega_2+2\omega_3-\omega_1)\,, \tag{19c}$$
$$\cos\gamma_1=(\omega_3-\omega_2)[\omega_1(2\omega_2+2\omega_3-\omega_1)]^{-1/2}\,,$$

from which one gets

$$\sin^2\theta_{23}=-\lambda(\omega_1,\omega_2,\omega_3)[4\omega_2\omega_3]^{-1}\,, \tag{20a}$$

$$\sin^2\gamma_1=-\lambda(\omega_1,\omega_2,\omega_3)[\omega_1(2\omega_2+2\omega_3-\omega_1)]\,, \tag{20b}$$

where

$$\lambda(\omega_1,\omega_2,\omega_3)=\omega_1{}^2+\omega_2{}^2+\omega_3{}^2-2\omega_2\omega_3-2\omega_1\omega_3-\omega_2\omega_3\,.$$

A remarkable property of γ_1 and θ_{23} is shown by Eqs. (20), namely, that $\sin\gamma_1$ and $\sin\theta_{23}$ vanish together. This result does not depend on the special choice of the masses, since it expresses only the fact that, when all $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are collinear, $\mathbf{q}_{23}$ is also collinear to them.

We now introduce a new set of variables in place of the components of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. First let us introduce the total momentum $\mathbf{P}$. Then, in the total center-of-mass system $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ add up to zero and make up a triangle with sides equal to p_1, p_2, and p_3. This triangle is completely defined, up to a displacement, by the lengths of its sides or, equivalently, by ω_1, ω_2, and ω_3. In order to fix the position of the triangle in space, it is useful to introduce a reference system of axes linked to it. We define that reference system as being right-handed, the z axis lying along $\mathbf{p}_i$ (which is one of the three vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, chosen once for all) and the y axis being normal to the plane of the triangle. Keeping the freedom of choice of the momentum alongside the z axis will help us to maintain a more symmetrical notation in the following. Finally, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ will be completely determined if one knows the three Euler angles[10] (ψ,θ,ϕ) which define the position of that body-fixed reference system with respect to a space-fixed reference system. Finally, the wave function will be a function of $\mathbf{P}$, ω_1, ω_2, ω_3, ψ, θ, and ϕ.

In fact, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are not only arguments of the wave function but they are also quantum numbers which completely label a state $|\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle$. We choose for the normalization of these states

$$\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle=\delta(\mathbf{p}_1'-\mathbf{p}_1)\delta(\mathbf{p}_2'-\mathbf{p}_2)\delta(\mathbf{p}_3'-\mathbf{p}_3)\,. \tag{21}$$

Another convenient complete set of commuting observables is provided by $\mathbf{P}_1$, ω_1, ω_2, ω_3, the square of the total angular momentum $\mathbf{J}^2=J(J+1)$, and the projection M of $\mathbf{J}$ on the body-fixed axis, together with its projection M_z on the space-fixed z axis. An eigenstate of this set of observables will be denoted by $|\mathbf{P},\omega_1,\omega_2,\omega_3,J,M,M_z\rangle$. As $\mathbf{P}$ is a constant of the motion, it will be convenient to put it equal to zero. In that case, we shall consider it no further, and write the state as $|\omega_1,\omega_2,\omega_3,J,M,M_z\rangle$. Furthermore, we shall often note all three symbols ω_1, ω_2, ω_3 by only one ω. Lastly, as $\mathbf{J}$ is a constant of the motion, and so is the space-fixed z axis, M_z will also be a constant of the motion and will appear only as a dummy index, so that we shall frequently omit it, writing simply the state: $|\omega JM\rangle$.

These states will be normalized according to

$$\begin{aligned}\delta(P')\langle\mathbf{P}',\omega_1',\omega_2',\omega_3',J',M',M_z'|\mathbf{P},\omega_1,\omega_2,\omega_3,J,M,M_z\rangle\\ =\delta(\mathbf{P})\delta(\mathbf{P}')\delta(\omega_1-\omega_1')\delta(\omega_2-\omega_2')\\ \times\delta(\omega_3-\omega_3')\delta_{JJ'}\delta_{MM'}\delta_{MzMz'}\,.\end{aligned} \tag{22}$$

The passage of the basis $|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle$ of the Hilbert space to the basis $|\mathbf{P},\omega,J,M,M_z\rangle$ will be completely determined by the coefficients $\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|\mathbf{P},\omega,J,M,M_z\rangle$. Because of the meaning of the different variables, one has necessarily

$$\begin{aligned}\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|\mathbf{P},\omega_1,\omega_2,\omega_3,J,M,Mz\rangle\\ =A\,\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\delta(\mathbf{P})\delta(\omega_1-\omega_1')\delta(\omega_2-\omega_2')\\ \times\delta(\omega_3-\omega_3')\mathfrak{D}_{M_zM}{}^{J*}(\psi,\theta,\phi)\,,\end{aligned} \tag{23}$$

where $\mathfrak{D}_{M_zM}{}^J(\psi,\theta,\phi)$ is the conventional rotation matrix which represents the rotation with Euler angles (ψ,θ,ϕ) in the irreducible representation of angular momentum J.[10] The coefficient A is a normalization coefficient which can be easily deduced from a comparison between Eqs. (21) and (22) as

$$A=[(2J+1)m_1{}^{-1}m_2{}^{-1}m_3{}^{-1}8^{-1}\pi^{-2}]^{1/2}\,. \tag{24}$$

Furthermore, the number of states in a domain of measure $d^3\mathbf{P}d\omega_1d\omega_2d\omega_3$, with J, M, and M_z fixed, is equal to

$$d^3Pd\omega_1d\omega_2d\omega_3\,. \tag{25}$$

Although the calculation of expressions (24) and (25) is straightforward, the necessary steps are indicated in the Appendix.

IV. THE INHOMOGENEOUS TERM OF THE FADEEV EQUATIONS

We have now to find what the Fadeev Eqs. (9) become when we take their matrix elements between states $|\mathbf{P}\omega JM\rangle$. For instance, one matrix element of the first row in the left-hand side of Eq. (9) will be $\langle\mathbf{P}'\omega'JM'|T^{(1)}(z)|\mathbf{P}\omega JM\rangle$. Taking advantage of the conservation of total momentum, we shall extract the δ functions which take care of it, by writing

$$\begin{aligned}\delta(\mathbf{P})\langle\mathbf{P}'\omega'JM'|T^{(1)}|\mathbf{P}\omega JM\rangle\\ =\delta(\mathbf{P})\delta(\mathbf{P}')\langle\omega'JM'|\mathcal{T}^{(1)}|\omega JM\rangle\,,\end{aligned} \tag{26a}$$

$$\begin{aligned}\delta(\mathbf{P})\langle\mathbf{P}'\omega'JM'|T_{23}|\mathbf{P}\omega JM\rangle\\ =\delta(\mathbf{P})\delta(\mathbf{P}')\langle\omega'JM'|\mathcal{T}_{23}|\omega JM\rangle\,,\end{aligned} \tag{26b}$$

and so on.

Our first task will be to compute (26b), which is the inhomogeneous term in the Fadeev equations. Using Eq. (23) and defining

$$\begin{aligned}\delta(p_1'+p_2'+p_3')\langle p_1',p_2',p_3'|T_{23}(z)|p_1,p_2,p_3\rangle\\ =\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\\ \times\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|\mathcal{T}_{23}(z)|p_1,p_2,p_3\rangle\,,\end{aligned} \tag{27}$$

[10] Our conventions for Euler angles and rotation matrices follow A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

one has

$$\delta(\mathbf{P}')\langle\mathbf{P}'\rangle\langle\mathbf{P}'\omega'JM'M_z'|T_{23}(z)|\mathbf{P}\omega JMM_z\rangle$$
$$=\int d^3p_1\cdots d^3\mathbf{p}_3'A^2\delta(\omega_1-E_1)\delta(\omega_3'-E_3')$$
$$\times\mathfrak{D}_{M_z'M'}{}^{J*}(R')\mathfrak{D}_{M_zM}{}^{J}(R)\delta(\mathbf{P}')\delta(\mathbf{P})\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)$$
$$\times\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\langle\mathbf{p}'|\mathcal{T}_{23}(z)|\mathbf{p}\rangle. \quad (28)$$

In writing that last equation we have tried to avoid unnecessary indices. Here R stands for the set of three Euler angles (ψ,θ,ϕ), while $E_1=p_1^2/2m_1$, and so on. According to Eq. (6), the matrix element in Eq. (28) is equal to

$$\langle\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|\mathcal{T}_{23}(z)|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle$$
$$=\langle\mathbf{p}_2',\mathbf{p}_3'|\hat{\mathcal{T}}_{23}(z-E_1)|\mathbf{p}_2,\mathbf{p}_3\rangle\delta(\mathbf{p}_1'-\mathbf{p}_1). \quad (29)$$

In order to compute expression (28), one can proceed as follows:

(i) integrate the two δ functions of $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3$ and $\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3'$ on $d^3\mathbf{p}_1$ and $d^3\mathbf{p}_1'$;

(ii) choose the body-fixed z axis along $\mathbf{p}_1$ so that

$$d^3\mathbf{p}_1d^3\mathbf{p}_3=m_1m_2m_3dE_1dE_2dE_3dR, \quad (30)$$

where dR is the measure on the rotation group,

$$dR=d\cos\theta d\psi d\phi;$$

(iii) rewrite the δ function in Eq. (29) as

$$\delta(\mathbf{p}_1-\mathbf{p}_1')=(m_1p_1)^{-1}\delta(E_1-E_1')$$
$$\times\delta(\cos\theta-\cos\theta')\delta(\psi-\psi'); \quad (31)$$

(iv) remark that

$$\langle\mathbf{p}_2'\mathbf{p}_3'|\hat{\mathcal{T}}_{23}(z-E_1)|\mathbf{p}_2\mathbf{p}_3\rangle$$
$$=F_{23}(E_1;E_2,E_3;E_2',E_3';u;z-E_1) \quad (32)$$

depends only on the angle

$$u=\phi-\phi', \quad (33)$$

so that after integration on $\cos\theta'$ and ϕ, using Eq. (31), we are left with a rotation matrix,[10]

$$\mathfrak{D}_{M_z'M'}{}^{J*}(\psi,\theta,\phi')=e^{iM'(\phi'-\phi)}\mathfrak{D}_{M_z'M'}{}^{J*}(\psi,\theta,\phi); \quad (34)$$

(v) replace the remaining integration over angles $d\cos\theta d\psi d\phi d\psi'$ by $d\cos\theta d\psi d\phi du$. The integration over the three first Euler angles can then be performed explicitly, using the orthogonality relation (A5) of the Appendix for the rotation matrices. Then expression (28) becomes

$$\delta(\mathbf{P})\delta(\mathbf{P}')m_1m_2m_3(m_1p_1)^{-1}\delta_{M'M}\delta_{M_zM_z'},$$
$$\int F_{23}(\omega_1;\omega_2\omega_3;\omega_2'\omega_3';u;z-\omega_1)\delta(\omega_1-\omega_1')e^{iMu}du. \quad (35)$$

Finally one has

$$\langle\omega'J'M_1'|\mathcal{T}_{23}(z)|\omega JM_1\rangle=m_1m_2m_3\delta_{M_1'M_1},$$
$$\int F_{23}(\omega,\omega',u,z-E_1)e^{iM_1u}du\delta(\omega_1-\omega_1')(m_1p_1)^{-1}, \quad (36)$$

where we have written M_1 rather than M to emphasize that Eq. (36) is true only when the body-fixed z axis is chosen alongside $\mathbf{p}_1$.

In order to remove that last condition, we have to find what Eq. (36) becomes when the body-fixed axis, while lying in the plane of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, is not collinear to $\mathbf{p}_1$ (for instance, when it is along $\mathbf{p}_2$, or along the bisection of the angle between $\mathbf{p}_2$ and $\mathbf{p}_3$—in any case, with a convention that is the same for the initial and the final states). At this point, we shall define as α_1 the angle between $0z$ and $\mathbf{p}_1$. Then one has[10]

$$|M_1\rangle=|M\rangle d_{MM_1}{}^J(-\alpha_1), \quad (37)$$

where the matrix $d_{MM_1}{}^J(\alpha)$ is defined as[10]

$$d_{MM_1}{}^J(\alpha)=\mathfrak{D}_{MM_1}{}^J(0,\alpha,0), \quad (38)$$

and therefore

$$\langle\omega'JM'|\mathcal{T}_{23}(z)|\omega JM\rangle$$
$$=m_1m_2m_3(m_1p_1)^{-1}\delta(\omega_1-\omega_1')$$
$$\times\int F_{23}(\omega,\omega',u,z-\omega_1)d_{M'M_1}{}^J(-\alpha_1')$$
$$\times\exp(iM_1u)d_{M_1M}(\alpha_1)du. \quad (39)$$

It is clear from Eq. (36) that the matrix in Eq. (39) is the rotation matrix for a rotation of an angle $-u$ around the axis $\mathbf{p}_1$ in the irreducible representation of angular momentum J.

Maybe some comments are in order at that stage concerning the function F_{23} and how to compute it. It is obvious from Eq. (29) that F_{23} is nothing but the off-the-energy-shell scattering amplitude for particles 2 and 3. Generally, this scattering amplitude is a function of four variables: the c.m. initial momentum q_{23}, the final c.m. momentum q_{23}', the extended c.m. energy ξ, and the cosine of the scattering angle $\cos\beta$. Let us write it $f_{23}(q_{23}^2,q_{23}'^2,\xi,\cos\beta)$. Then we have

$$F_{23}(\omega_1;\omega_2,\omega_3;\omega_2',\omega_3';u;z-E_1)$$
$$=f_{23}[q_{23}^2,q_{23}'^2,m_{23}Q_{23}^2(2m_2m_3)^{-1};$$
$$\cos\gamma_1\cos\gamma_1'+\sin\gamma_1\sin\gamma_1'\cos u], \quad (40)$$

where q_{23}^2 is given by Eq. (19); $q_{23}'^2$ by an analogous expression, $\cos\gamma_1$, is given by Eq. (17); while

$$m_{23}Q_{23}^2(2m_2m_3)^{-1}=z-Mm_{23}^{-1}\omega_1. \quad (41)$$

In practical applications, it will be useful to express the scattering amplitude in terms of the partial-wave amplitudes by

$$F_{23}=\sum_l(2l+1)a_l(q_{23}^2,Q_{23}^2,q_{23}'^2)$$
$$\times P_l(\cos\gamma,\cos\gamma_1'+\sin\gamma_1\sin\gamma_1'\cos u). \quad (42)$$

Many of the applications in which one can be interested are concerned with the case in which only one amplitude is substantially different from zero in the series (42), this amplitude having a resonance at an energy E with width Γ. With these conditions, (42) becomes

$$F_{23}=\frac{A(\mathbf{q}_{23})B(\mathbf{q}_{23}')}{z-Mm_{23}^{-1}\omega_1-E+i(\Gamma/2)}. \tag{43}$$

For further consideration concerning the meaning of the quantities entering into Eq. (43) and their relations to the wave functions of the resonating state, see Lovelace.[9]

V. REDUCED FADEEV EQUATIONS

The complete Fadeev equations can be reduced in the same way we used for the inhomogeneous term. One thus obtains

$$\mathcal{T}_{M'M}{}^{(i)J}(\omega',\omega)=\mathcal{T}_{klM'M}{}^{J}(\omega',\omega)-\int K_{M'M''}{}^{(i,j)J}(\omega',\omega'')\mathcal{T}_{M''M}{}^{(j)J}(\omega'',\omega)d\omega'' \tag{44}$$

$(i,\ j,\ k,\ l=1,\ 2,\ 3,\ i\neq k,\ i\neq l,\ k\neq l)$, and

$$\mathcal{T}_{M'M}{}^{(i)J}(\omega',\omega)=\langle\omega_1',\omega_2',\omega_3',J,M'|\,\mathcal{T}_{(z)}{}^{(i)}\,|\omega_1,\omega_2,\omega_3,J,M\rangle, \tag{45}$$

$$\mathcal{T}_{klM'M}{}^{J}(\omega',\omega)=m_1m_2m_3(2\pi)^{-9}(m_ip_i)^{-1}\delta(\omega_i-\omega_i')\int F_{kl}(\omega,\ \omega',\ u,\ z-\omega_i)d_{M'M_i}{}^{J}(-\alpha_i')\times e^{iM_iu}d_{M_iM}{}^{J}(\alpha_i)du, \tag{46}$$

$$K_{M'M''}{}^{(i,j)J}(\omega',\omega'')=0 \quad\text{for}\quad i=j, \tag{47a}$$

$$K_{M'M''}{}^{(i,j)J}(\omega',\omega'')=m_1m_2m_3(m_ip_i)^{-1}\delta(\omega_i'-\omega_i'')\times\int F_{kl}(\omega',\ \omega'',\ z-\omega_i',\ u)d_{M'M_i}{}^{J}(-\alpha_i')\times e^{iM_iu}d_{MM''}{}^{J}(\alpha_i'')[\omega_1''+\omega_2''+\omega_3''-z]^{-1}du \tag{47b}$$

for $i\neq j$. (Here $k\neq i$, $l\neq i$, $k\neq l$.)

In fact, the kernel (47) is not completely continuous, since it still contains δ functions. However, its square $(K^J)^2$ is completely continuous. This result follows immediately from the proof by Fadeev that K^2 is completely continuous and

$$\mathrm{Trace}K^2K^{2\dagger}=\sum_J(2J+1)\ \mathrm{Trace}(K^J)^2(K^{J\dagger})^2, \tag{48}$$

which shows that

$$\mathrm{Trace}(K^J)^2(K^{J\dagger})^2<\infty, \tag{49}$$

i.e., that K^2 is completely continuous.

As an application of this result, it is possible to approximate K^2 by finite matrices, i.e., if we replace the integration on ω_1'', ω_2'', ω_3'' by a summation on a finite set of values of these variables, the solution of the corresponding matrix problem tends to the solution of the operator problem when the number of values of the variables tends to infinity. Analogous considerations could be made for the Weinberg kernel.

Although the kernel K is not itself completely continuous, we believe that it is worth while to try to use it directly by solving the Fadeev equations directly into the form (44). The reason is that, owing to the δ function in Eq. (47b), the integration $d\omega''$ in Eq. (44) bears upon only two variables. This is an important simplification, when one wants to solve the equations on a computer, with respect to the use of K^2 where it is necessary to make triple integrations.

VI. CONCLUSIONS

We have indicated how to separate angular momentum in the Fadeev equations in a symmetrical way. The resulting equations will be used for a study of the problem of overlapping resonances where the approximation (43) for the two-body scattering amplitudes can be made. One problem to which this method is particularly well suited is to find what effects the spins of two resonances can have on the Dalitz plot in the region where the two resonances overlap.[11] Another interesting type of problem to study is whether three strong two-body interactions can generate a three-body resonance. Such an effect has been hinted at in the $K\bar{K}\pi$ system,[12] where both $K\pi$ and $\bar{K}\pi$ show a K^* resonance while the low-energy S-wave K-$\bar{K}$ interaction is presumably strong. While our method is essentially nonrelativistic, it is easy to derive relativistic approximation which, at least, will keep the qualitative character of the interactions.

The following paper is devoted to the extension of Eq. (44) to complex values of the angular momentum.

Finally, it may be worth while to indicate that, at least in principle, the present method can be extended to systems of more than three particles.

APPENDIX

In this Appendix, we indicate how to compute the coefficient A of Eq. (23) and the density of states (25).

One first computes the scalar product (22) as

$$\int d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{p}_3\delta(\mathbf{P}')\langle\mathbf{P}'\omega'JM'M_z'|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle\times\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|\mathbf{P}\omega JMM_z\rangle, \tag{A1}$$

then replaces the scalar products in that expression ac-

[11] C. Bouchiat and G. Flamant, Nuovo Cimento **23**, 13 (1962).

[12] R. Armenteros, D. N. Edwards, T. Jacobsen, A. Shapira, J. Vandermeulen *et al.*, communication to the Sienna International Conference on Elementary Particles, October 1963 (unpublished).

cording to Eq. (23), thus getting

$$[A^2]\delta(\mathbf{P})\delta(\mathbf{P}')\int\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\delta(\omega_1-p_1^2/2m_1)\cdots \times\delta(\omega_3-p_3'^2/2m_3)\mathfrak{D}_{M_z'M_z'}{}^{J*}(\psi,\theta,\phi) \times\mathfrak{D}_{M_zM}{}^{J}(\psi,\theta,\phi)d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{p}_3. \quad \text{(A2)}$$

One evaluates (A2) in the following way:

(1) Dispose of the δ function by integrating over $\mathbf{p}_3$.

(2) Choose a system of body-fixed axes with the z axis in the direction of $\mathbf{p}_1$, and $\mathbf{p}_2$ in the plane of the x and z axes. Then

$$d^3\mathbf{p}_1d^3\mathbf{p}_2=p_1^2p_2^2dp_1dp_2d\cos\theta_{12}\sin\theta_{12}d\psi d\theta d\phi. \quad \text{(A3)}$$

(3) Using the expression analogous to Eq. (16) for $\cos\theta_{12}$, pass from (A3) to

$$d^3\mathbf{p}_1d^3\mathbf{p}_2=m_1m_2m_3d\omega_1d\omega_2d\omega_3\sin\theta d\psi d\theta d\phi. \quad \text{(A4)}$$

(4) Using the orthogonality property of the rotation matrices,[10]

$$\int\mathfrak{D}_{M_zM}{}^{J}(\psi,\theta,\phi)\mathfrak{D}_{M_z'M'}{}^{J*}(\psi,\theta,\phi)\sin d\psi d\theta d\phi =[8\pi^2/(2J+1)]\delta_{JJ'}\delta_{MM'}\delta_{M_zM_z'}, \quad \text{(A5)}$$

one gets an expression of the scalar product which can be directly compared to Eq. (22) and gives for the coefficient A the expression (24).

Let us now compute the number of states with fixed values of J, M, M_z, in the domain $d^3\mathbf{P}d\omega$, $d\omega_2d\omega_3$. Let us start from the development of the δ function on the rotation group in terms of rotation matrices,

$$\delta(\cos\theta)\delta(\psi)\delta(\phi)=\sum_{Jm}[8\pi^2/(2J+1)]\mathfrak{D}_{mm}{}^{J}(\psi,\theta,\phi). \quad \text{(A6)}$$

Equation (A6) can be obtained by using the orthogonality property (A5).

Clearly the total momentum will give a factor d^3p for the number of states. Around $\mathbf{P}'=0$, the number of states will be of the form $X_J(\omega_1,\omega_2,\omega_3)d\omega_1d\omega_2d\omega_3$. In order to compute the function $X_J(\omega_1,\omega_2,\omega_3)$, we evaluate the scalar product (21), which leads to

$$\langle\mathbf{p}_1{}^1,\mathbf{p}_2{}^1,\mathbf{p}_3{}^1|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) =\int\sum_{JMM'}\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')d\omega_1d\omega_2d\omega_3d^3\mathbf{P}X_J(\omega_1\omega_2\omega_3) \times\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|\mathbf{P}\omega_1\omega_2\omega_3JMM'\rangle \times\langle JMM'\omega_1\omega_2\omega_3\mathbf{P}|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle; \quad \text{(A7)}$$

when Eqs. (23) and (24) are used, this integral is

$$\sum_{JMM'}A^2\int\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')d\omega_1d\omega_2d\omega_3d^3\mathbf{P} \times\delta(\omega_1-p_1^2/2m_1)\cdots\delta(\omega_3'^2-p_3'^2/2m_3) \times\mathfrak{D}_{M'M}{}^{J*}(R')\mathfrak{D}_{M'M}{}^{J}(R)X_J(\omega_1\omega_2\omega_3), \quad \text{(A8)}$$

where R stands for (ψ,θ,ϕ). Then one clearly has

$$\sum_{JMM'}X_J\mathfrak{D}_{MM'}{}^{J*}(R')\mathfrak{D}_{MM'}{}^{J}(R) =\sum_{JM}X_J\mathfrak{D}_{MM}{}^{J}(R'^{-1}R). \quad \text{(A9)}$$

Equation (A6) suggests putting

$$X_J(\omega_1\omega_2\omega_3)=[(2J+1)/8\pi^2]X(\omega_1\omega_2\omega_3), \quad \text{(A10)}$$

so that (A8) is simply

$$A^2\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3') \times\delta(P_1^2/2m_1-P_1'^2/2m_1)\cdots \times\delta(p_3^2/2m_3-P_3^2/2m_3)\delta(\bar{R}'R) \times(P_1^2/2m_1,P_2^2/2m_2,P_3^2/2m_3) =\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2')\delta(\mathbf{p}_3-\mathbf{p}_3')\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3). \quad \text{(A11)}$$

In order to fix $X(\omega_1,\omega_2,\omega_3)$ one integrates (A11) over $d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{p}_3$, using Eq. (A3), which reads

$$\int\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{p}_3 =m_1m_2m_3d\left(\frac{P_1^2}{2m_1}\right)\cdots d\left(\frac{P_3^2}{2m_3}\right)dR,$$

which leads immediately to the expression (25) for the density of states.